

Post-quantum Sigma Protocols and Signatures from **Low-Rank Matrix Completions**

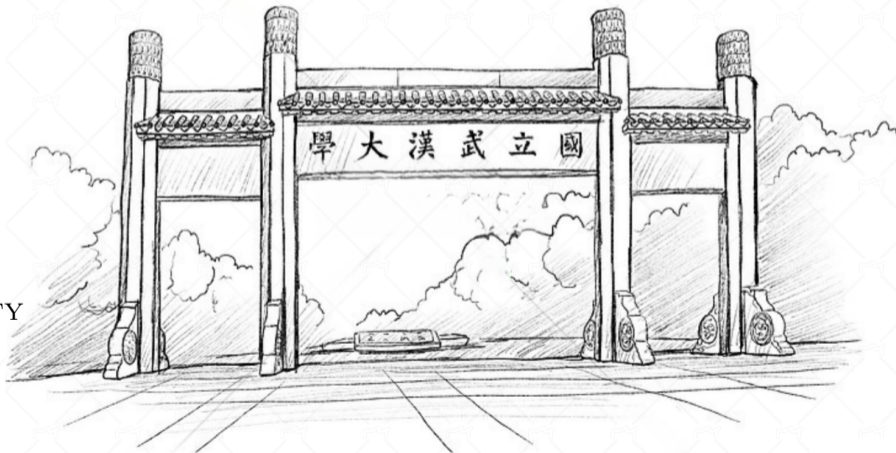
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PROVSEC 2023



Motivation

Designing Provable and Practical Post-quantum Signature Schemes **NOT** from Lattices

1. Digital Signature:

- Message integrity and identity authentication.
- Quantum Computers \Rightarrow Need to be resistant to classical/quantum adversaries.
- Schemes standardized by the NIST in July 2022:
 - { Dilithium and Falcon: **Structural Lattices-based**, unknown new attack.
 - { SPHINCS+: **Hash-based**, larger sizes.

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2. Demand in diversity \Rightarrow NIST's new call for proposals in Sept 2022.



Moody, Dustin (Fed)

to pqc-forum

7 Sept 2022, 4:16:00 am



All,

From pqc-forum

NIST is calling for additional digital signature proposals to be considered in the PQC standardization process. NIST is primarily interested in **additional general-purpose signature schemes that are not based on structured lattices**. For certain applications, such as certificate transparency, NIST may also be interested in signature schemes that have short signatures and fast verification. NIST is open to receiving additional submissions based on structured lattices, but is intent on diversifying the post-quantum signature standards. As such, any structured lattice-based signature proposal would need to significantly outperform CRYSTALS-Dilithium and FALCON in relevant applications and/or ensure substantial additional security properties to be considered for standardization.

Contribution

Sigma Protocols and Signature Schemes from LRMC

We found there is a hard problem in **linear algebra**, named

Low-Rank Matrix Completions (LRMC)

could be used in **post-quantum crypto designing** $\left\{ \begin{array}{l} \text{Sigma Protocol} \\ \text{Signature Scheme – NIST's call} \end{array} \right.$

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1

Goal: Completing the left to the right (low-rank, e.g., $rank = 1$).

Contribution

Sigma Protocols and Signature Schemes from LRMC

In short, we present

1. A **Sigma Protocol** from LRMC, with soundness error $2/3$
2. The first protocol + Sigma Protocol with Helper [Beu20]
= LRMC-based **Sigma Protocol**, with soundness error $1/2$
3. The second protocol + recent techniques + Fiat-Shamir Transformation [FS86]
= LRMC-based **Signature Scheme**, with competitive sizes and simple settings

Outline

1 Preliminaries

- ▶ Preliminaries
- ▶ LRMC-based Sigma Protocols (with Helper)
- ▶ LRMC-based Signature Scheme

Hard Problems

MinRank and 1-MinRank

MinRank Problem

Input: An integer r , and $s + 1$ matrices $\mathbf{M}_0; \mathbf{M}_1, \dots, \mathbf{M}_s \in \text{Mat}_{k,l}(\mathbb{F})$

Output: $\alpha_1, \dots, \alpha_s \in \mathbb{F}$, such that

$$\text{rank}(\mathbf{M}_0 + \sum_{i=1}^s \alpha_i \mathbf{M}_i) \leq r$$

Features:

- NP-Complete, and hard for random instances \Rightarrow post-quantum (details later).
- Simple: Based on linear algebra computations, and has high efficiency.
- Extensively studied: Cryptanalysis of Rainbow, GeMSS, HFE/HFEv-, etc.
- **1-MinRank Problem**: a special case, requires the rank of $\mathbf{M}_1, \dots, \mathbf{M}_s$ are 1.

Hard Problems

Low-Rank Matrix Completion (LRMC)

Low-Rank Matrix Completion Problem

Input: An integer r , and a matrix $\mathbf{M} \in \text{Mat}_{k,l}(\mathbb{F})$ with s unfilled entries

Output: $\alpha_1, \dots, \alpha_s \in \mathbb{F}$, such that completing the remaining to a matrix with $\text{rank} \leq r$

- A toy example, $r = 2, k = 3, l = 4, \mathbb{F} = \mathbb{F}_7$:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & * & * \\ * & * & 5 & 2 \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 4 & 3 \\ 1 & 3 & 5 & 2 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 1 & 6 \\ 3 & 6 & 5 & 2 \end{bmatrix}$$

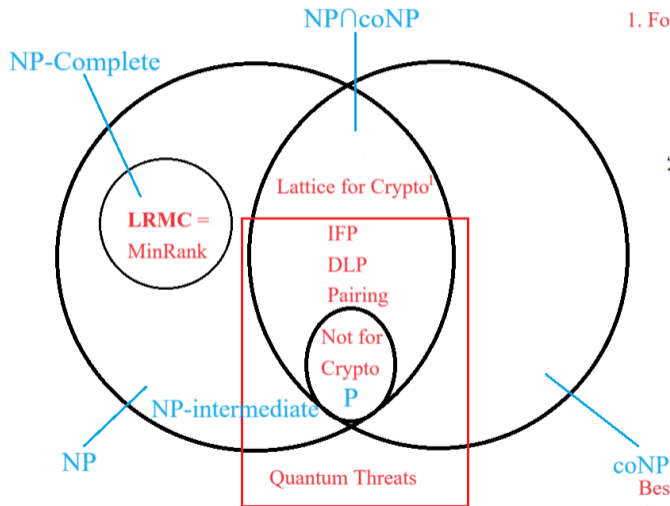
We have $\text{rank}(\mathbf{M}_1) = 3 > 2$, $\text{rank}(\mathbf{M}_2) = 1 \leq 2$, and $(1, 6, 3, 6)$ is a solution.

- Equivalence [Der18]: “The instances can be mutual transformed.”

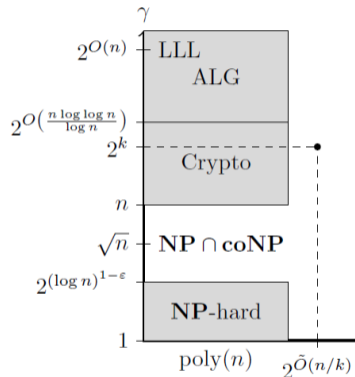
$$\text{MinRank} \Leftrightarrow 1\text{-MinRank} \Leftrightarrow \text{LRMC}$$

Hard Problems

The Complexity Comparisons



1. For standard lattices, we have SVP_γ



From Vinod Vaikuntanathan's slides in CS294, UC Berkeley

Besides, standard lattices are more reliable than structural lattices (Dilithium and Falcon)

Building Blocks

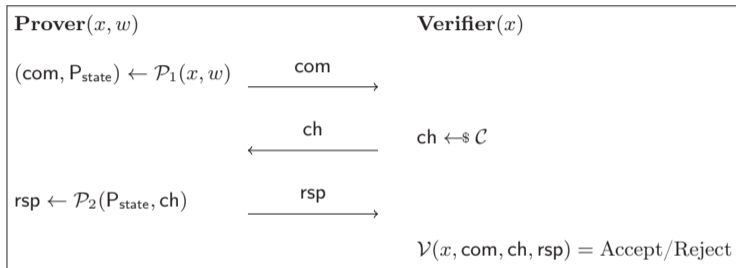
3-move Sigma Protocol [CD95]



Ronald Cramer



Ivan Damgård

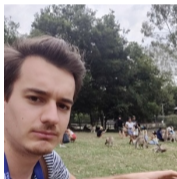


Goal: Prove the knowledge w such that $(x, w) \in \mathcal{R}$

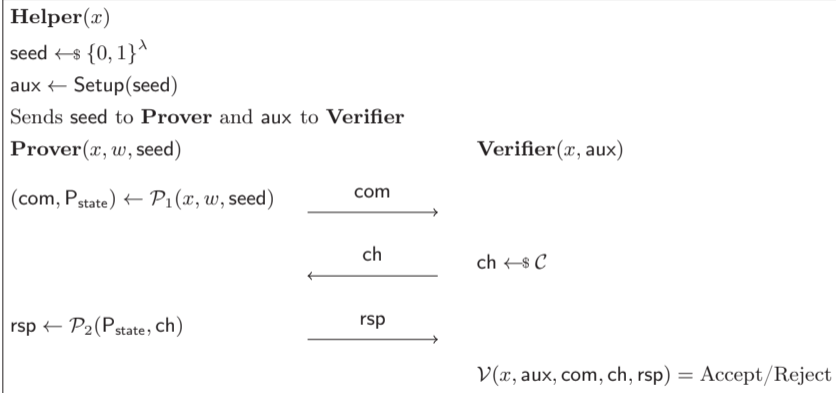
- Completeness
- Soundness
- Special Honest-Verifier Zero-Knowledge (SHVZK)

Building Blocks

From 3-move Sigma Protocol to Sigma Protocol with Helper [Beu20]



Ward Beullens



Helper: Trusted by the **Prover** and the **Verifier**.

Goal: Prove the knowledge w such that $(x, w) \in \mathcal{R}$.

Building Blocks

From 3-move Sigma Protocol to Sigma Protocol with Helper [Beu20]

Goal: Prove the knowledge w such that $(x, w) \in \mathcal{R}$.

Security properties of Sigma Protocol with Helper:

- **Completeness.** **Prover** holds w is always Accepted.
- **2-Special Soundness.** The witness w can be efficiently extracted, i.e., there exists a **polynomial-time knowledge extractor** \mathcal{E} to use two valid transcripts

$(x, \text{aux}, \text{com}, \text{ch}, \text{rsp})$ and $(x, \text{aux}, \text{com}, \text{ch}', \text{rsp}')$.

- Soundness error p : Any efficient adversary $\mathcal{A}(1^\lambda, x)$ passes the protocol with prob. $\leq p + \text{negl}(\lambda)$.
- **HV Zero-Knowledge.** There exists a **polynomial-time simulator** $\mathcal{S}(x)$ that produces transcripts indistinguishable from ones by **Prover** (x, w) .

Building Blocks

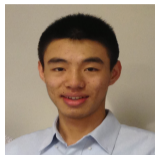
Removing the Helper – Using Cut-and-choose technique [KKW18] to simulate the Helper



Jonathan Katz



Vladimir Kolesnikov



Xiao Wang

Prover(x, w)

For $i \in \{1, \dots, s\}$:

$\text{seed}_i \leftarrow_{\$} \{0, 1\}^\lambda$ $\xrightarrow{\text{com}_i, \text{aux}_i, \forall i}$

$\text{aux} \leftarrow \text{Setup}(\text{seed}_i)$ $\xleftarrow{I, \{\text{ch}_i\}_{i \in I}}$

com_i and rsp_i as before $\xrightarrow{\{\text{seed}_i\}_{i \notin I}, \{\text{rsp}_i\}_{i \in I}}$

Verifier(x)

Samples $I \subset \{1, \dots, s\}, |I| = \tau$

Checks $\text{aux}_i = \text{Setup}(\text{seed}_i), \forall i \notin I$

Validates $(x, \text{aux}_i, \text{com}_i, \text{ch}_i, \text{rsp}_i)_{i \in I}$

Building Blocks

From Sigma Protocol to Signature Scheme – Using Fiat-Shamir Transformation [FS86]



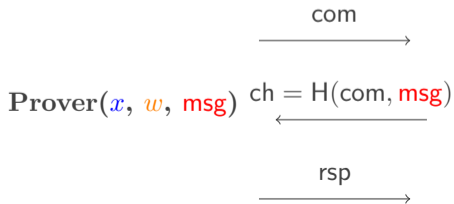
Amos Fiat



Adi Shamir



You should remember the generic construction, even you've been ashes.



$\implies (\text{com}, \text{rsp})$ is a signature for **msg**

The transcripts of the simulated protocol

Outline

2 LRMC-based Sigma Protocols (with Helper)

▶ Preliminaries

▶ LRMC-based Sigma Protocols (with Helper)

▶ LRMC-based Signature Scheme

Our Sigma Protocol

How to obtain a LRMC instance

Prover obtains a LRMC instance for crypto construction, as follows:

1. **Prover** chooses a random matrix $\mathbf{A} = (a_{i,j}) \leftarrow \text{Mat}_{k,l}(\mathbb{F})$, s.t. $\text{rank}(\mathbf{A}) = r$;
2. **Prover** removes s entries $(a_{i_1,j_1}, a_{i_2,j_2}, \dots, a_{i_s,j_s})$ to obtain a partially filled matrix \mathbf{A}^- .

Let the public key is \mathbf{A}^- , and the witness is $(a_{i_1,j_1}, a_{i_2,j_2}, \dots, a_{i_s,j_s}) = (a_{i_t,j_t})$ for $1 \leq t \leq s$. Then, the completed matrix \mathbf{A} is a solution for the LRMC Problem.

Recall: Low-Rank Matrix Completion Problem

Input: An integer r , and a matrix $\mathbf{M} \in \text{Mat}_{k,l}(\mathbb{F})$ with s unfilled entries

Output: $\alpha_1, \dots, \alpha_s \in \mathbb{F}$, such that completing the remaining to a matrix with $\text{rank} \leq r$

Our Sigma Protocol

How to design a Zero-Knowledge Protocol to prove the Relation – Hiding the witness

Prover proves a relation $(x, w) = (\mathbf{A}^-, a_{i_1, j_1}, a_{i_2, j_2}, \dots, a_{i_s, j_s})$ in zero-knowledge, as follows:

1. **Prover** breaks it into $\mathbf{A}^- = \mathbf{A}_1^- + \mathbf{A}_2^-$, where $\mathbf{A}_1^-, \mathbf{A}_2^-$ are also partially filled, s.t. \mathbf{A}_1^- is 1 at its filled entries, and divides $a_{i_t, j_t} = \alpha_{i_t, j_t} + \beta_{i_t, j_t}$ for $1 \leq t \leq s$;

$$\underbrace{\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & * & * \\ * & * & 5 & 2 \end{bmatrix}}_{\mathbf{A}^-} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & * & * \\ * & * & 1 & 1 \end{bmatrix}}_{\mathbf{A}_1^-} + \underbrace{\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 3 & * & * \\ * & * & 4 & 1 \end{bmatrix}}_{\mathbf{A}_2^-}$$

2. **Prover** completes \mathbf{A}_1^- (\mathbf{A}_2^-) with α_{i_1, j_1} (β_{i_1, j_1}) to obtain \mathbf{A}_1 (\mathbf{A}_2), i.e.,

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \alpha_{i_1, j_1} & \alpha_{i_2, j_2} \\ \alpha_{i_3, j_3} & \alpha_{i_4, j_4} & 1 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 3 & \beta_{i_1, j_1} & \beta_{i_2, j_2} \\ \beta_{i_3, j_3} & \beta_{i_4, j_4} & 4 & 1 \end{bmatrix}$$

We have $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$.

Our Sigma Protocol

How to design a Zero-Knowledge Protocol to prove the Relation – Stern-like framework [Ste93]

Based on: $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \Rightarrow \mathbf{PAQ} = (\mathbf{PA}_1\mathbf{Q} + \mathbf{Y}) + (\mathbf{PA}_2\mathbf{Q} - \mathbf{Y}), \forall \mathbf{P}, \mathbf{Q}, \mathbf{Y}$

Prover($x = \mathbf{A}^-$, $w = (a_{i_t, j_t})$)

Verifier($x = \mathbf{A}^-$)

$\mathbf{c}_0 := \text{Com}(r_0, \mathbf{P}, \mathbf{Q}, \mathbf{Y})$

$\mathbf{c}_1 := \text{Com}(r_1, \mathbf{PA}_1\mathbf{Q} + \mathbf{Y})$

$\text{com} := (\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2)$
→

$\mathbf{c}_2 := \text{Com}(r_2, \mathbf{PA}_2\mathbf{Q} - \mathbf{Y})$

← ch $\text{ch} \leftarrow_{\$} \{0, 1, 2\}$

If $\text{ch} = 0$, then reveals $\mathbf{c}_1, \mathbf{c}_2$

$\text{rsp} := (r_1, r_2, \mathbf{PA}_1\mathbf{Q} + \mathbf{Y}, \mathbf{PA}_2\mathbf{Q} - \mathbf{Y})$

If $\text{ch} = 1$, then reveals $\mathbf{c}_0, \mathbf{c}_2$

$\text{rsp} := (r_0, r_2, \mathbf{P}, \mathbf{Q}, \mathbf{Y}, \beta = (\beta_{i_t, j_t}))$

→ rsp *Accpet/Reject?*

If $\text{ch} = 2$, then reveals $\mathbf{c}_0, \mathbf{c}_1$

$\text{rsp} := (r_0, r_1, \mathbf{P}, \mathbf{Q}, \mathbf{Y}, \alpha = (\alpha_{i_t, j_t}))$

Our Sigma Protocol with Helper

How to decrease the soundness error from $2/3$ to $1/2$ [Beu20]

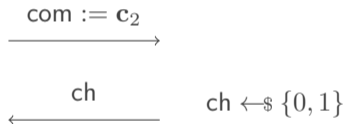
Based on: $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \Rightarrow \mathbf{PAQ} = (\mathbf{PA}_1\mathbf{Q} + \mathbf{Y}) + (\mathbf{PA}_2\mathbf{Q} - \mathbf{Y}), \forall \mathbf{P}, \mathbf{Q}, \mathbf{Y}$

Helper: $\mathbf{P}, \mathbf{Q}, \mathbf{Y} \leftarrow \text{PRG}(\text{seed}), \mathbf{c}_0 := \text{Com}(r_0, \mathbf{P}, \mathbf{Q}, \mathbf{Y}), \mathbf{c}_1 := \text{Com}(r_1, \mathbf{PA}_1\mathbf{Q} + \mathbf{Y})$

Prover($x = \mathbf{A}^-, w = (a_{i_t, j_t}), \text{seed}$)

Verifier($x = \mathbf{A}^-, (\mathbf{c}_0, \mathbf{c}_1)$)

$\mathbf{c}_2 := \text{Com}(r_1, \mathbf{PA}_1\mathbf{Q} + \mathbf{Y})$



If $\text{ch} = 0$, then reveals $\mathbf{c}_1, \mathbf{c}_2$

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If $\text{ch} = 1$, then reveals $\mathbf{c}_0, \mathbf{c}_2$

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Our Sigma Protocol with Helper

Further Optimizations [Beu20, BESV22]

We take similar tricks to optimize the sizes [Beu20, BESV22], includes:

- Using **cut-and-choose technique** [KKW18] to drop the pre-processing, and remove the helper.
- Using **Merkle Tree** to compress and recompute the commitments.
- Using **Binary Tree** to optimize the transmission of seeds.
- Using several **MPC tricks** to improve parallel repetitions.

Our Sigma Protocol with Helper

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Pierre de Fermat

“I have a proof of the theorem, but there is not enough space in this margin.”– Pierre de Fermat

Please check our paper for more details :)

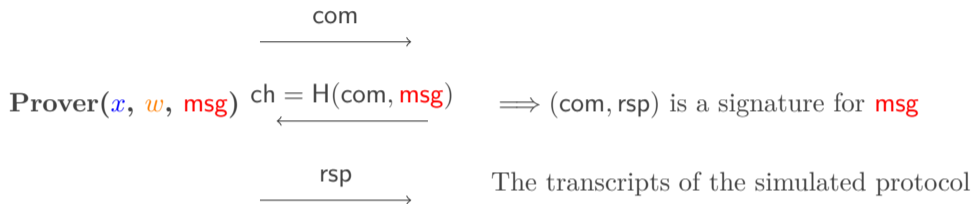
Outline

3 LRMC-based Signature Scheme

- ▶ Preliminaries
- ▶ LRMC-based Sigma Protocols (with Helper)
- ▶ LRMC-based Signature Scheme

Our Signature Scheme

From Sigma Protocol to Signature Scheme – Using Fiat-Shamir Transformation [FS86]



To sign on the message μ , **Signer** executes the following steps:

1. Runs **the first move** of the Sigma Protocol to generate **com** ;
2. Computes $\text{ch} = \text{H}(\text{com}, \mu)$;
3. Runs **the third move** of the Sigma Protocol to obtain **rsp**, and the signature on μ is (com, rsp) .

Our Signature Scheme

Parameters: [BESV22] (PQCrypto'22) is the state-of-the-art, [Cou01] (AC'01) is the pioneering

Parameter Set		I	II	III
λ [Security parameter]		128	192	256
q [Order of finite field $\mathbb{F} = \mathbb{F}_q$]		16	16	16
(k, l) [Dimensions of matrix \mathbf{A}]		(14, 14)	(17, 17)	(20, 20)
r [Rank of matrix \mathbf{A}]		4	6	6
s [Unfilled number of matrix \mathbf{A}]		108	130	208
Public Key Size (B)	This work	44	80	96
	[BESV22]	60 (by seeds)	104 (by seeds)	128 (by seeds)
	[Cou01]	114 (by seeds)	169 (by seeds)	232 (by seeds)
Signature Size (KB)	This work	24	54	97
	[BESV22]	24	54	97
	[Cou01]	55	118	221
PK + Sig Storage (KB)	This work	24	54	97
	[BESV22]	34	72	137
	[Cou01]	65	136	261

Our Signature Scheme

Comparisons

- **Comparing with MinRank-based schemes** [[BESV22](#), [Cou01](#)].
 - **Storage-Lower**: LRMC-based avoid seeds for the PK generation, leading a significant reduction in **total storage costs of the public key and signature** when actual signing, e.g., more than 30% for the Parameter Set I.
 - **Time-Shorter**: LRMC-based avoid linear combinations and matrix-vector multiplications between hundreds of matrices $\in \text{Mat}_{k,l}(\mathbb{F}_q)$ to recover the PK, saving considerable time.
 - **Conceptually-Simpler**: LRMC-based are more intuitive and succinct, only one (partially completed) matrix $\mathbf{A} \in \text{Mat}_{k,l}(\mathbb{F}_q)$ in the system parameters, instead of $s + 1$ matrices in the same dimensions.
- **Comparing with NIST Standards.**
 - SPHINCS+: Sizes are in the same magnitude, e.g., $\lambda = 128$
 - 44B vs 32B in Public Key Size
 - 24KB vs 17KB in Signature Size
 - Dilithium and Falcon: The underlying hard problem LRMC is NP-Complete, providing stronger security guarantee than problems over Structural Lattices.

Retrospect and Prospect

Following the Research Philosophy of Modern Cryptography [GSC⁺23]

Conclusions: In this work, we present

1. A new NP-Complete problem – LRMC, for crypto designing.
2. A 3-move ZK proof for the solution of LRMC.
3. Decreasing the soundness error from $2/3$ to $1/2$.
4. A signature scheme from LRMC.

Future work:



Self-Cultivation

- {
- New Construction: { ZK protocol for LRMC with smaller error.
Trapdoor LRMC-based signatures.
 - New Foundation:** New hard problems with better sizes/efficiencies.
 - New Definition: Formalizing new primitives, and instantiating them from assumptions (LRMC, Lattice, Pairing).
- }

Q&A

Thank you for listening!

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